Repetitive higher cluster categories of type A_n

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Abstract

We show that the repetitive higher cluster category of type A_n, defined as the orbit category $\mathcal{D}^b(\text{mod}kA_n)/(\tau^{-1}[m])^p$, is equivalent to a category defined on a subset of diagonals in a regular $p(nm+1)$-gon. This generalizes the construction of Caldero-Chapoton-Schiffler, [CCS06], which we recover when $p = m = 1$, and the work of Baur-Marsh, [BM08], treating the case $p = 1, m > 1$. Our approach also leads to a geometric model of the bounded derived category $\mathcal{D}^b(\text{mod}kA_n)$.

1 Introduction

In this paper we give a geometrical-combinatorial model, in the spirit of Caldero-Chapoton-Schiffler, [CCS06], of repetitive higher cluster categories of type A_n. These are orbit categories of the bounded derived category of modkA_n under the action of the cyclic group generated by the auto-equivalence $(\tau^{-1}[m])^p$, where kA_n is the path algebra associated to a Dynkin quiver of type A_n, τ is the AR-translation and $[m]$ the composition of the shift functor [1] on $\mathcal{D}^b(\text{mod}kA_n)$ with itself m-times and $1 \leq p \in \mathbb{N}$. We write

$$\mathcal{C}^m_{n,p} := \mathcal{D}^b(\text{mod}kA_n)/ < (\tau^{-1}[m])^p > .$$

The class of objects is given by orbits of objects in the derived category under the autoequivalence $(\tau^{-1}[m])^p$, and the space of morphisms is as follows

$$\text{Hom}_{\mathcal{C}^m_{n,p}}(X, Y) = \bigoplus_{i\in\mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod}kA_n)}(X, (\tau^{-p}[pm])^i Y).$$

When the index p or m is equal to one we omit it in the writing of $\mathcal{C}^m_{n,p}$.

When $p = m = 1$ one recovers the usual cluster categories, which we denote simply by $\mathcal{C}_n$, defined independently in [CCS06] for the case A_n and in the general case in [BMR$^+$06]. If $p = 1$ and $m > 1$ one regains the higher cluster categories $\mathcal{C}^m_n$, also called m-cluster categories, for which Baur-Marsh gave a geometric model in [BM08] and [BM07]. In the case $p > 1$ and $m = 1$, the category is simply called repetitive cluster category $\mathcal{C}_{n,p}$, studied also by Zhu, from a purely algebraic point of view in [Zhu08].

The main results of the paper are the following. On one side we are able to give an equivalence of categories between $\mathcal{C}^m_{n,p}$ and a category defined on a subset of all the diagonals in a regular $p(nm+1)$-gon Π^p. Then, the model we propose here also leads to a geometric interpretation of cluster tilting objects in $\mathcal{C}^m_{n,p}$. By this we mean a maximal collection of pairwise non-isomorphic indecomposable

objects $T_1, \dots T_t$ such that $\mathrm{Ext}^i(T_j, T_k) = 0$, for $1 \leq i \leq m$ and $1 \leq j, k \leq t$. Furthermore, for $m = 1$ we are able to prove an equivalence of triangulated categories between $\mathcal{C}_{n,p}$ and a quotient of a cluster category of a certain rank. On the other hand as an application of the results obtained for $\mathcal{C}^m_{n,p}$, a geometric model for $\mathcal{D}^b(\mathrm{mod}kA_n)$ will be given.

The association of geometric models to algebraic categories has been studied and developed by many authors, among others we mention: [BM08], [BM07], [BZ10], [CCS06], [BT09], [Tor11],[Sch08],.... This approach is not only beautiful but also fruitful as it gives new ways to understand the intrinsic combinatorics of the category.

The particularity of the repetitive higher cluster categories is that they are fractionally Calabi-Yau of dimension $\frac{p(m+1)}{p}$, this means that $(\tau[1])^p \cong [p(m+1)]$ as triangle functors, and the fraction cannot be simplified. And it is precisely in this point that the category $\mathcal{C}^m_{n,p}$ differs from $\mathcal{C}_n$ and $\mathcal{C}^m_n$.

This generalization of the notion of a Calabi-Yau category is interesting because many categories happen to be of this type. Consider for example the bounded derived category of $\mathrm{mod}kQ$, for a quiver Q with underlying Dynkin diagram, or of the category of coherent sheaves on an elliptic curve or a weighted projective line of tubular type. A.-C. van Roosmalen was recently able to give a classification up to derived equivalence of abelian hereditary categories whose bounded derived category are fractionally Calabi-Yau, see [van10].

The structure of the paper is as follows. The next section is dedicated to a review of useful definitions on fractionally Calabi-Yau categories and repetitive cluster categories will be carefully defined. In Section 3 we define the repetitive polygon Π^p and we choose a subset of diagonals in Π^p together with a rotation rule between them. This will lead to the modelling of $\mathcal{C}_{n,p}$. In Section 4 we study the relation between repetitive cluster categories and cluster categories. Section 5 is dedicated to the study of cluster tilting objects and complements to almost complete basic cluster tilting objects in $\mathcal{C}_{n,p}$. The content of Section 6 is the geometric modelling of $\mathcal{C}^m_{n-1,p}$, the construction we give here generalizes the one of Section 3. Finally, in Section 7 we extend the construction of Π^p to a geometric figure with an infinite number of sides, $\Pi^{\pm\infty}$. Applying the results of Section 7 to the category generated by the set of diagonals complementary to 2-diagonals in $\Pi^{\pm\infty}$, we obtain a model for $\mathcal{D}^b(\mathrm{mod}kA_n)$.

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2 Repetitive cluster categories of type A_n

2.1 Serre duality and Calabi-Yau categories

Let k be a field (assume it to be algebraically closed, even though in this section it is not needed) and let $\mathcal{K}$ be a k-linear triangulated category which is Hom-finite, i.e. for any two objects in $\mathcal{T}$ the space of morphisms is a finite dimensional vector space.

Notation 2.1. *Throughout the paper we let $\mathcal{D}$ be $\mathcal{D}^b(\mathrm{mod}kA_n)$ or $\mathcal{D}^b(\mathrm{mod}kA_{n-1})$, and we believe that it will be clear from the context if it is one or the other.*

Definition 2.2. *A* Serre functor *for a k-triangulated category $\mathcal{K}$ is an auto-equivalence $\nu : \mathcal{K} \to \mathcal{K}$ together with bifunctorial isomorphisms*

$$D\mathrm{Hom}_{\mathcal{K}}(X,Y) \cong \mathrm{Hom}_{\mathcal{K}}(Y, \nu X),$$

for each $X, Y \in \mathcal{K}$. D indicates the vector space duality $\mathrm{Hom}_k(?, k)$.

We will say that $\mathcal{K}$ has Serre duality if $\mathcal{K}$ admits a Serre functor. In the case $\mathcal{K} = \mathcal{D}$ a Serre functor exists ([Kel10] p. 24), it is unique up to isomorphism and $\nu \xrightarrow{\sim} \tau[1]$, where τ is the Auslander-Reiten translate and [1] the shift functor of $\mathcal{D}$.

Definition 2.3. *A* triangle functor *between two triangulated categories $\mathcal{J}$ and $\mathcal{K}$ is a pair (F, σ) where $F : \mathcal{J} \to \mathcal{K}$ is a k-linear functor and $\sigma : F[1] \to [1]F$ an isomorphism of functors such that the image of a triangle in $\mathcal{J}$ under F is a triangle in $\mathcal{K}$.*

Suppose (F, σ) and (G, γ) are triangle functors, then a morphism of triangle functors *is a morphism of functors $\alpha : F \to G$ such that the square*

$$\begin{tikzcd} F[1] \arrow[r, "\sigma"] \arrow[d, "\alpha[1]"] & {[1]F} \arrow[d, "{[1]\alpha}"] \\ G[1] \arrow[r, "\gamma"] & {[1]G} \end{tikzcd}$$

commutes.

Definition 2.4. *One says that a category $\mathcal{K}$ with Serre functor ν is a* fractionally Calabi-Yau category of dimension $\frac{m}{n}$ *or* $\frac{m}{n}$-Calabi-Yau *if there is an isomorphism of triangle functors:*

$$\nu^n \cong [m],$$

for $n, m > 0$, and where $[m]$ indicates the composition of the shift functor with itself m times.

Remark 2.5. *Notice that a category of fractional CY dimension $\frac{m}{n}$ is also of fractional CY dimension $\frac{mt}{nt}$, $t \in \mathbb{Z}$. However the converse is not always true.*

2.2 Repetitive cluster categories of type A_n

In the following we are interested in giving a geometric description of the orbit category of the bounded derived category of modkA_n under the action of the cyclic group generated by the auto-equivalence $(\tau^{-1}[1])^p = \tau^{-p}[p]$ for $p > 0$. Where τ is the AR-translation in $\mathcal{D}$ and [1] the shift functor.

Definition 2.6. *The* repetitive cluster category

$$\mathcal{C}_{n,p} := \mathcal{D}/ < \tau^{-p}[p] >$$

of type A_n, has as class of objects the $\tau^{-p}[p]$-orbits of objects in $\mathcal{D}$. The class of morphism is given by:

$$\mathrm{Hom}_{\mathcal{C}_{n,p}}(X,Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, (\tau^{-p}[p])^i Y)$$

Observe that when $p = 1$, one gets back the usual cluster category which we simply denote by $\mathcal{C}_n$.

As next we summarize some basic properties of $\mathcal{C}_{n,p}$ proven in [Zhu08, Proposition: 3.3].

Lemma 2.7.

- *$\mathcal{C}_{n,p}$ is a triangulated category with AR-triangles and Serre functor $\nu := \tau[1]$.*
- *The projections $\pi_p : \mathcal{D} \to \mathcal{C}_{n,p}$ and $\eta_p : \mathcal{C}_{n,p} \to \mathcal{C}_n$ are triangle functors.*
- *$\mathcal{C}_{n,p}$ is fractionally CY of dimension $\frac{2p}{p}$.*
- *$\mathcal{C}_{n,p}$ is a Krull-Schmidt category.*
- $\mathrm{Ind}(\mathcal{C}_{n,p}) = \bigcup_{i=0}^{p-1} \mathrm{Ind}((\tau^{-1}[1])^i \mathcal{C}_n)$.

Remark 2.8. *The full subcategory* $\mathrm{Ind}(\mathcal{C}_n)$ *of* $\mathcal{C}_n$ *is a (usually not full) subcategory of* $\mathrm{Ind}(\mathcal{C}_{n,p})$.

In the following let $\mathcal{F}^1 = \mathcal{F}$ be the fundamental domain for the functor $\tau^{-1}[1]$ in $\mathcal{D}$ given by the representatives of the isoclasses of objects in $\mathrm{Ind}(\tau^{-1}[1])(\mathcal{C}_n)$. That is isoclasses of indecomposables in $\mathrm{mod}(kA_n)$ together with the [1]-shift of the projective indecomposable modules. Let $F^k := F \circ \cdots \circ F$, k-times, then we denote by $\mathcal{F}^k$ the F^k-shift of $\mathcal{F}$ and we can draw the fundamental domain for the functor $\tau^{-p}[p]$ as in Figure 1.

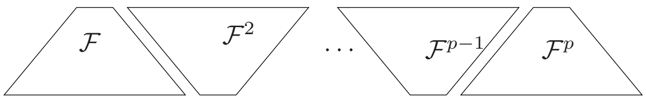

Figure 1: Partition of the fundamental domain of $\tau^{-p}[p]$ for an odd value of p

3 Geometric model of $\mathcal{C}_{n,p}$

3.1 The repetitive polygon Π^p

Let $p > 1$. For the purpose of the geometric characterization of $\mathcal{C}_{n,p}$ let Π be a regular $N := n + 3$-gon and let Π^p be a regular $p(n + 2)$-gon. Number the vertices of Π^p clockwise repeating p-times the $N-$tuple $1, 2, ..., N - 1, N$ and letting correspond $N \equiv 1$. Then we denote by Π_1 a region homotopic to Π inside Π^p delimited by the segments $(1, 2), (2, 3) \ldots, (N - 1, N)$ and the inner arc $(1, N)$.

Denote by $\rho : \Pi^p \to \Pi^p$ the clockwise rotation through $\frac{2\pi}{p}$ around the center of Π^p, and set $\Pi_k := \rho^{k-1}(\Pi_1)$ for $1 \leq k \leq p$. In this way we divide Π^p into p regions. See Figure 2 where we illustrate this construction.

Definition 3.1. *We call* diagonals *of* Π^p *the union of all diagonals of*

$$\Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_p.$$

Denote the diagonals of Π^p by the triple (i,j,k), where $1 \leq k \leq p$ specifies a region Π_k inside Π^p, and the tuple (i,j) defines the diagonal in Π_k.

Notice that for us the set of diagonals of Π^p consists of a subset of all the straight inner lines (drawn as arcs) joining the vertices of Π^p. Furthermore, the arcs $(1,N,k)$ for $1 \leq k \leq p$ are not diagonals of Π^p as they correspond to boundary segments of Π_k.

Notation 3.2. *Observe once and for all that in the writing (i,j,k) we understand that the index k has to be taken modulo p, and the indices i,j modulo N. Furthermore, we always assume that $i < j$.*

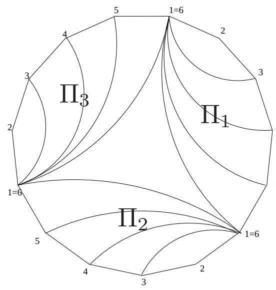

Figure 2: The polygon Π^3 for $n = 3$

3.2 Quiver of diagonals of Π^p

As next we associate a translation quiver to the diagonals of Π^p with the intention of modelling the AR-quiver of the category $\mathcal{C}_{n,p}$. The first part of the next definition goes back to [CCS06], the second part is new and essential for the modelling of the repetitive cluster category.

In the following we call a rotation of a diagonal around a fixed vertex *irreducible* whenever the other vertex of the diagonal moves to the preceding or successive vertex.

Definition 3.3. *Let $\Gamma_{n,p}$ be the quiver whose vertices are the diagonals (i,j,k) of Π^p, and whose arrow are defined as follows.*

1. *For $1 \leq i < j < N$:*

$$\begin{tikzcd}[row sep=small] & (i,j+1,k) \\ (i,j,k) \arrow[ur] \arrow[dr] & \\ & (i+1,j,k) \end{tikzcd}$$

That is, we draw an arrow if there is an irreducible clockwise rotation around the vertex j or i inside Π_k,

2. *For $1 \leq i < j = N$ we link $(i,N,k) \to (i+1,N,k)$ whenever there is an irreducible clockwise rotation around the vertex N inside Π_k*

3. *For $1 \leq i < j = N$ we link $(i,N,k) \to (1,i,k+1)$. That is, we draw an arrow to the diagonal we can reach with a clockwise rotation around vertex i inside Π_k composed with ρ. I.e. (i,N,k) is linked to $\rho(1,i,k) = (1,i,k+1)$.*

The set of the operations 1. *and* 2. *are denoted by* IrrRot *and the operations* 3. *by* IrrρRot.

Notice that $\Gamma_{n,p}$ lies on a Möbius strip when p is odd, and on a cylinder when the value of p is even.

As next we equip $\Gamma_{n,p}$ with a translation $\tau : \Gamma_{n,p} \to \Gamma_{n,p}$ such that $(\Gamma_{n,p}, \tau)$ becomes a stable translation quiver in the sense of Riedtmann, [Rie80].

For the reader familiar with the article of [CCS06] we point out that the first part of the following definition agrees with the one given there, however the second is new and can be thought of as a connecting map between the different regions Π_k in Π^p.

Definition 3.4. *The* translation τ *on* $\Gamma_{n,p}$ *maps*

- (i, j, k) *to* $(i-1, j-1, k)$ *for* $1 < i, j \leq N$ *and* $1 \leq k \leq p$, *i.e.* τ *is an anticlockwise rotation around the center of* Π^p *by* $\frac{2\pi}{p(n+2)}$.
- $(1, j, k)$ *to* $(j-1, N, k-1)$, *i.e. the effect of* τ *on* $(1, j, k)$ *is induced by* $\frac{1}{N}$*th anticlockwise rotation in the regular* N*-gon* Π *homotopic to* Π_k, *composed with* ρ^{-1}. *That is*

$$(1, j, k) \mapsto (j-1, N, k) \mapsto (j-1, N, k-1) = \rho^{-1}(j-1, N, k).$$

Lemma 3.5. *The pair* $(\Gamma_{n,p}, \tau)$ *is a stable translation quiver.*

Proof. It is clear that the map τ is bijective. As $\Gamma_{n,p}$ is finite, we only need to persuade us that the number of arrows from a diagonal D to D' is equal to the number of arrows from $\tau D'$ to D. As there is at most one arrow between any two diagonals, we only have to check that there is an arrow from D to D' if and only if there is an arrow from $\tau D'$ to D.

For a given diagonal $D = (i, j, k)$ we distinguish two possible cases. Either there is an arrow to a diagonal D' of Π_k to a diagonal of the neighboring region Π_{k+1}.

- In the first case we distinguish depending on whether $i = 1$, or $i \neq 1$. If $i \neq 1$ the result follows from Proposition 2.2 in [BM08]. If $i = 1$, then

$$\begin{tikzcd} & (1, j+1, k) \\ (1, j, k) \arrow[ur] \arrow[dr] & \\ & (2, j, k) \end{tikzcd}$$

  Then $\tau(1, j+1, k) = (j, N, k-1)$, and $(j, N, k-1) \to (1, j, k)$. Similarly $\tau(2, j, k) = (1, j-1, k)$ and $(1, j-1, k) \to (1, j, k)$.

- If there is an arrow $(i, j, k) \to (i', j', k+1)$, we deduce that

$$j = N, i' = 1, \text{ and } j' = i \neq 1.$$

  Then $\tau(1, i, k+1) = (i-1, N, k)$, and we deduce that

$$(i-1, N, k) \to (i, N, k).$$

$\square$

3.3 Category of diagonals in Π^p

We now associate an additive category $\mathcal{C}(\Pi^p)$ to the diagonals of Π^p. The new feature arising here is that we only allow a subset of all possible diagonals of Π^p, and that we deal with an additional type of rotation.

As next we remind the reader the definition of mesh category. This will be used repeatedly in the next sections.

Let α denote the arrow $x \to y$, then $\sigma(\alpha)$ denotes the arrow $\tau(y) \to x$.

Definition 3.6. *The* mesh category *of a translation quiver (Q, τ) is the factor category of the path category of Q modulo the ideal generated by the* mesh relations

$$r_v := \sum_{\alpha: u \to v} \alpha \cdot \sigma(\alpha),$$

where the sum is over all arrows ending in v, and v runs through the vertices of Q.

Then the category of diagonals $\mathcal{C}(\Pi^p)$ in Π^p arises as the mesh category of $(\Gamma_{n,p}, \tau)$. More specifically, the class of objects of $\mathcal{C}(\Pi^p)$ is given by formal direct sums of the diagonals in Π^p, i.e. diagonals in the regions Π_k for $1 \leq k \leq p$. The class of morphisms is generated by the two rotations IrrRot_{Π^p} and $\mathrm{Irr}\rho\mathrm{Rot}_{\Pi^p}$, carefully defined in Definition 3.3, modulo the mesh relations which can be read off from $(\Gamma_{n,p}, \tau)$.

Note that with our approach a new geometric type of mesh relation appears. It arises between diagonals of two consecutive regions: Π_k and Π_{k+1}.

3.4 Equivalence of categories

The next results will show that the category $\mathcal{C}(\Pi^p)$ of diagonals in Π^p is equivalent to the repetitive cluster category of type A_n. The notation is as specified after Lemma 2.7.

Proposition 3.7. *There is a bijection*

$$\varphi : X \mapsto D_X,$$

from $\mathrm{Ind}(\mathcal{C}_{n,p})$ to the diagonals of Π^p such that:

- *$Irr_{\mathcal{C}_{n,p}}(X, Y) \longleftrightarrow \mathrm{IrrRot}_{\Pi^p}(D_X, D_Y)$ if X, Y are in $\mathcal{F}^i$, for some i.*
- *$Irr_{\mathcal{C}_{n,p}}(X, Y) \longleftrightarrow \mathrm{Irr}\rho\mathrm{Rot}_{\Pi^p}(D_X, D_Y)$ if $X \in \mathcal{F}^i$ and $Y \in \mathcal{F}^j$, $i \neq j$.*

Proof. We saw in Lemma 2.7 that

$$\mathrm{Ind}(\mathcal{C}_{n,p}) = \bigcup_{i=1}^{p} \mathrm{Ind}(\mathcal{F}_i).$$

Then one easily sees that φ is a bijection on the level of objects. In fact, we can apply [CCS06, Corollary 4.7] to every subcategory $\mathrm{Ind}(\mathcal{F}_i)$ of $\mathcal{C}_{n,p}$ and to the corresponding region Π_i in Π^p, $1 \leq i \leq p$.

Then for each $1 \leq i \leq p$, the irreducible morphisms between objects X, Y in $\mathcal{F}_i$ where $X \ncong (\tau^{-1}[1])^i P[1]$ for P some projective, or where

$$X \cong (\tau^{-1}[1])^i P[1] \text{ and } Y \cong (\tau^{-1}[1])^i Q[1],$$

for projectives P and Q, agree by [CCS06, Theorem 5.1] with the rotations $\mathrm{IrrRot}_{\Pi_i}(D_X, D_Y)$ described in part 1. and 2. of Definition 3.3.

Therefore, we only need to study the correspondence between $\mathrm{Irr}_{\mathcal{C}_{n,p}}(X, Y)$ and the rotations in Π^p described in part 3. of Definition 3.3 for

$$X \cong ((\tau^{-1}[1])^i)\widetilde{P}[1] \text{ and } Y \ncong (\tau^{-1}[1])^i \widetilde{Q}[1].$$

Thus, assume $\mathrm{Irr}_{\mathcal{C}_{n,p}}(X, Y) \neq 0$, in this case it follows from the shape of the AR-quiver of $\mathcal{C}_{n,p}$ that $Y \cong (\tau^{-1}[1])^{i+1} R$, for some projective R. Under the bijection φ applied to the objects X and Y one then has:

$$X \mapsto (i, N, k), \qquad Y \mapsto (1, i, k+1).$$

From Definition 3.3 it then follows that there is an arrow

$$(i, N, k) \to (1, i, k+1)$$

in $\Gamma_{n,p}$ defining a corresponding operation in $\mathrm{Irr}\rho\mathrm{Rot}_{\Pi^p}(D_X, D_Y)$. Similarly one shows the other direction.

This shows that the mapping φ is a bijection also on the level of morphisms.

It only remains to check that the mesh relations in the two categories agree. But by the above we have precisely showed that the AR-quiver of $\mathcal{C}_{n,p}$ and $\Gamma_{n,p}$ are isomorphic, hence the mesh relations in both categories agree. $\square$

It follows from the previous result that the projection functor $\eta_p : \mathcal{C}_{n,p} \twoheadrightarrow \mathcal{C}_n$ corresponds to the projection $\mu_p : \Pi^p \to \Pi$.

4 $\mathcal{C}_{n,p}$ and the link to the cluster category

The desire of comparing the category $\mathcal{C}_{n,p}$ with the cluster category of type A_t for a certain t arose while studying the geometrical model of $\mathcal{C}_{n,p}$. We will see in this section that the embedding is slightly different than expected, as t cannot simply be deduced from the size of Π^p. This is because the class of morphisms in $\mathcal{C}_{n,p}$ is too particular.

However, starting with a bigger polygon, the category $\mathcal{C}_{n,p}$ appears whenever we assume that $p \neq 2$. In fact, otherwise the diagonals of the two models overlap hence the description degenerates.

Lemma 4.1. *Let $p > 2$. Then $\mathcal{C}_{n,p}$ is a subcategory of $\mathcal{C}_t$ for a suitable value of t. In fact,*

- *for p even set $t := (n+3)(\frac{p}{2}) - 3$,*
- *for p odd set $t := (n+3)p - 3$.*

As a first observation we point out that the subcategories are not full. Throughout the proof we will adopt the following notation. We denote by $(\Gamma_{n,p}, \tau_{n,p})$ the AR-quiver of $\mathcal{C}_{n,p}$ and by (Γ_t, τ_t) the AR-quiver of $\mathcal{C}_t$.

Proof. To prove the claim it is enough to establish an isomorphism of stable translation quivers between $(\Gamma_{n,p}, \tau_{n,p})$ and a stable translation subquiver of (Γ_t, τ_t) for the different cases.

Let us first study the case where p is even. We observe that the union of the τ_t-orbits of the top and bottom n rows of the quiver Γ_t, illustrated in the gray strip in Figure 3, defines a subquiver $\tilde{\Gamma}_t$ of Γ_t. Since $p > 2$, the top and the bottom n rows do not overlap.

As Γ_t is a stable translation quiver, the same remains true for $\tilde{\Gamma}_t$. And it turns out that when t is as given in the claim, $\tilde{\Gamma}_t$ and $\Gamma_{n,p}$ are isomorphic as stable translation quivers. First: by construction, the two quivers have the same number of rows (namely n). To see the isomorphism just on the level of quivers we compare the induced action of the auto equivalence $(\tau_{n,p}^{-1}[1])^p$ on $\Gamma_{n,p}$ with the action of $(\tau_t^{-1}[1])|_{\tilde{\Gamma}_t}$ on $\tilde{\Gamma}_t$: It is easy to check that the actions coincide. Furthermore, because the meshes of the quivers $\tilde{\Gamma}_t$ and $\Gamma_{n,p}$ coincide we deduce that this gives an isomorphism of stable translation quivers.

Considering the mesh categories of $\tilde{\Gamma}_t$ and Γ_t itself we get that $\operatorname{Ind}(\mathcal{C}_{n,p})$ is a subcategory of $\operatorname{Ind}(\mathcal{C}_t)$ by Proposition 3.7. Then we extend the result to the whole category $\mathcal{C}_{n,p}$ simply by linearity.

When p is odd, the embedding of $\Gamma_{n,p}$ into Γ_t can be shown in a similar way. However, in this case we need to embed $\Gamma_{n,p}$ into the central band of Γ_t to preserve the induced action of the autoequivalence, i.e. to preserve the identifications of vertices in $\Gamma_{n,p}$. More precisely, this time we have to consider the horizontal band n vertices wide at the high of $\frac{(p-1)(n+3)}{2}+1$ vertices, counting from the bottom of Γ_t, see Figure 3. Then we concludes as in the previous case. □

4.1 Triangulated equivalence for $\mathcal{C}_{n,p}$

Since the inclusion of $\mathcal{C}_{n,p}$ in the cluster category $\mathcal{C}_t$ for t as in Lemma 4.1 does not give rise to a full subcategory, $\mathcal{C}_{n,p}$ is not a triangulated subcategory of $\mathcal{C}_t$. However, it is possible to prove that it is triangulated equivalent to a quotient category of $\mathcal{C}_t$.

Quotient categories of the type we are going to use here have been studied in [Jør10], et al. . In particular, Jørgensen showed that higher cluster categories are triangulated equivalent to quotients of the cluster category. We will now recall the definition of quotient categories and key properties, which will be needed in the following.

Let $\mathcal{C}$ be an additive category and $\mathbb{X}$ a class of objects of $\mathcal{C}$. Then the quotient category $\mathcal{C}_{\mathbb{X}}$ has by definition the same objects as $\mathcal{C}$, but the morphism spaces are taken modulo all the morphisms factoring through an object of $\mathbb{X}$.

Observe that if $\mathcal{C}$ is a triangulated category, then $\mathcal{C}_{\mathbb{X}}$ needs not to be triangulated for all choices of $\mathbb{X}$. However, it was shown in [Jør10, Theorem 2.2] that $\mathcal{C}_{\mathbb{X}}$ is always pre-triangulated. This means that $\mathcal{C}_{\mathbb{X}}$ is equipped with a pair (σ, ω) of adjoint endofunctors satisfying a number of properties. However none of them is necessarily an auto-equivalence. Nevertheless, taking a particular choice of the class $\mathbb{X}$, Jørgensen proved in [Jør10, Theorem 3.3] that the endofunctors σ and ω, can be turned into auto-equivalences, so that $(\mathcal{C}_{\mathbb{X}}, \sigma)$ becomes a triangulated category. We use these facts to link the triangulated structure of $\mathcal{C}_{n,p}$ to a quotient category of the cluster category of type A_t.

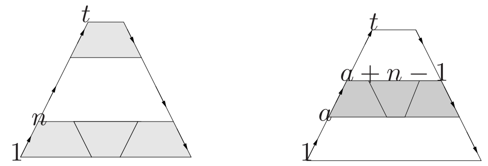

Figure 3: Inclusions $\mathcal{C}_n^4 \subset \mathcal{C}_t$ and $\mathcal{C}_n^3 \subset \mathcal{C}_t$, and $a = \frac{(p-1)(n+3)}{2} + 1$.

Proposition 4.2. *Assume $p \neq 2$ and define t as in Lemma 4.1. Then the category $\mathcal{C}_{n,p}$ is triangulated equivalent to a quotient of a cluster category of type A_t.*

Proof. For p even, denote by $\mathbb{X}$ the additive subcategory generated by the indecomposable objects in the band of $t - 2n$ vertices in the middle of Γ_t, for $t = (n+3)\frac{p}{2} - 3$. For p odd i.e. $t = (n+3)p - 3$ one takes as $\mathbb{X}$ the additive subcategory generated by the indecomposable objects in the band of $\frac{t-n}{2}$ vertices at the top and bottom of Γ_t. Then, in both cases we have that $\tau_t \mathbb{X} = \mathbb{X}$. So the AR-quiver of the quotient category $(\mathcal{C}_t)_{\mathbb{X}}$ is obtained by deleting the vertices corresponding to the objects of $\mathbb{X}$ and the arrows linked with them ([Jør10, Theorem 4.2]). Furthermore, $(\mathcal{C}_t)_{\mathbb{X}}$ is connected, and has finitely many indecomposable objects up to isomorphism. Furthermore, again by [Jør10, Theorem 4.2] $(\mathcal{C}_t)_{\mathbb{X}}$ is standard and of algebraic origin. Proceeding as in [Jør10, Theorem 5.2] we conclude that $(\mathcal{C}_t)_{\mathbb{X}}$ is triangulated equivalent to a quotient of a cluster category of type A_n.

It remains to see that $(\mathcal{C}_t)_{\mathbb{X}}$ and $\mathcal{C}_{n,p}$ are equivalent as triangulated categories. For this we observe that $\mathcal{C}_{n,p}$ is of algebraic origin by results of [Kel10, Section 9.3] and standard by [Ami07, Proposition 6.1.1.]. Furthermore, it is straightforward to see that the AR-quivers of $(\mathcal{C}_t)_{\mathbb{X}}$ and $\mathcal{C}_{n,p}$ are isomorphic as translation quivers. Thus we are in the conditions of Amiot's Theorem [Jør10, Theorem 5.1] applied to $(\mathcal{C}_t)_{\mathbb{X}}$ and $\mathcal{C}_{n,p}$. Hence we deduce that these categories are equivalent as triangulated categories, and so the claim follows. □

5 Cluster tilting theory for $\mathcal{C}_{n,p}$

It is well known that cluster tilting objects in cluster categories are strongly linked to clusters in cluster algebras. This link has been established in [BMR+06]. Studying the endomorphism algebra of these objects provides a way to recover the exchange matrix, indispensable in the mutation process in a cluster algebra.

In this section we are interested in understanding the cluster tilting objects of $\mathcal{C}_{n,p}$, and compare them with configurations of diagonals in the polygon Π^p. When $p = 1$, it is known that cluster tilting objects correspond to the formal direct sum of diagonals triangulating a regular $n+3$-gon.

Cluster tilting objects in $\mathcal{C}_{n,p}$ have also been studied in [Zhu08] from an algebraic point of view. In fact, Zhu proves that the endomorphism algebra of a cluster tilting object of $\mathcal{C}_{n,p}$ gives a cover of the endomorphism algebra of the same object, seen now as a cluster tilting object of the cluster category.

Definition 5.1. *A* cluster tilting set $\mathcal{T}$ *in $\mathcal{C}_{n,p}$ is a set of pairwise non isomorphic indecomposables objects, such that* $\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(T', T) = 0$ *for all $T, T' \in \mathcal{T}$,*

and $\mathcal{T}$ is maximal with this property. An object $T \in \mathcal{C}_{n,p}$ is called a cluster tilting object *if* $\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(T,T) = 0$ *and T has a maximal number of pairwise non isomorphic direct summands.*

A cluster tilting object is said to be *basic* if the direct sum is taken over all its objects in a cluster tilting set $\mathcal{T}$.

Lemma 5.2. *Cluster tilting sets in $\mathcal{C}_n$ correspond to maximal collections of non crossing diagonals in a regular $n+3$-gon. The cardinality of such a set is n.*

Proof. This follows from the fact that $\mathrm{Ext}^1_{\mathcal{C}_n}(T,T') \neq 0$ if and only if the diagonals corresponding to T and T' cross (where T, T' are arbitrary indecomposable objects in $\mathcal{C}_n$). See [CCS06, Section 5]. □

As our goal is to link cluster tilting objects of $\mathcal{C}_{n,p}$ to diagonals in Π^p we first need to express what it means geometrically that two objects have no extension. We will see that it is not enough to study only diagonals in a single region Π_i in Π^p, but also diagonals in the regions Π_{i-1} respectively Π_{i+1}, depending on which entry of the bifunctor $\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(-,-)$ is fixed.

We remind the reader that diagonals of Π^p consist only of a subset of all the lines joining vertices of Π^p. In the next result as well as in Corollary 5.4, we assume that $i < j$ and $i' < j'$ in the writing of the diagonals $D_X := (i,j,l)$ and $D_Y := (i',j',l')$ of Π^p.

Lemma 5.3. *Let $X, Y \in \mathrm{Ind}(\mathcal{C}_{n,p})$ and let $D_X := (i,j,l)$ and $D_Y := (i',j',l')$ be the corresponding diagonals of Π^p. Then*

$$\dim(\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(X,-)) = 1,$$

if and only if the diagonal D_y satisfies one of the following conditions

- *D_X, D_Y cross inside Π_l, i.e. $l = l'$, and $1 \leq i' < i < j' < j \leq N$;*
- *$\rho(D_Y)$ crosses D_X inside Π_l, i.e. $l = l' + 1 \,\mathrm{mod} N$, and $1 \leq i < i' < j < j' \leq N$.*

Otherwise $\dim(\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(X,-)) = 0$.

Proof. When $p = 1$, this was remarked in [CCS06, Section 5]. For all the other values of p we proceed using the fact that the triangulated category $\mathcal{C}_{n,p}$ has Serre duality by Lemma 2.7. Thus,

$$\begin{aligned} D\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(X,-) &= D\mathrm{Hom}_{\mathcal{C}_{n,p}}(X,-[1]) \\ &\cong \mathrm{Hom}_{\mathcal{C}_{n,p}}(-[1],\tau X[1]) \\ &\cong \mathrm{Hom}_{\mathcal{C}_{n,p}}(-,\tau X). \end{aligned}$$

Here we used that $\mathcal{C}_{n,p}$ has finite dimensional Hom spaces, and therefore also finite dimensional Ext spaces. As next we compute the $\mathrm{Hom}(-,\tau X)$ support using the mesh relations in the AR-quiver of $\mathcal{C}_{n,p}$. That gives a rectangular region, also known as the backwards hammock. Then one concludes using the bijection between indecomposable objects and diagonals from Proposition 3.7. □

Dually we have

Corollary 5.4. *Under the assumptions from Lemma 5.3*

$$\dim(\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(-,Y)) = 1$$

if and only if D_X satisfies one of the following conditions

- *D_X, D_Y cross inside $\Pi_{l'}$, i.e. $l = l'$, and $1 \leq i' < i < j' < j \leq N$,*
- *$\rho^{-1}(D_X)$ crosses D_Y inside $\Pi_{l'}$, i.e. $l = l'+1$ and $1 \leq i < i' < j < j' \leq N$,*

Otherwise $\dim(\mathrm{Ext}^1_{\mathcal{C}_{n,p}}(-,Y)) = 0$.

With these results in mind we are able to deduce the desired geometric description of cluster tilting sets, hence basic cluster tilting objects, of $\mathcal{C}_{n,p}$.

Recall that the map $\rho : \Pi^p \to \Pi^p$ was defined as a clockwise rotation of $\frac{2\pi}{p}$ around the center of Π^p.

Proposition 5.5. *Cluster tilting sets $\mathcal{T}'$ in $\mathcal{C}_{n,p}$ correspond bijectively with the collections*

$$\mathcal{T}' = \mathcal{T} \cup \rho(\mathcal{T}) \cup \ldots \rho^{p-1}(\mathcal{T})$$

where $\mathcal{T}$ is a triangulation of the region Π_1 in Π^p.

Proof. We saw in Lemma 5.3 and Corollary 5.4 that the non-vanishing of the Ext spaces also arises from diagonals in the neighboring regions Π_{k+1} and Π_{k-1} of Π^p. In particular, diagonals D_X in Π_k and D_Y in Π_{k+1} give rise to a non-vanishing Ext spaces if and only if $\rho^{-1}(D_Y)$ crosses D_X inside Π_k and the vertices of the diagonals satisfy the conditions of Corollary 5.4 (here X and Y are switched). Thus, the only possible way to get a maximal configuration of diagonals which does not cross in the sense of Lemma 5.3 and Corollary 5.4 is to take the same triangulation in each region Π_k, for $1 \leq k \leq p$.

On the other hand, it follows from Lemma 5.2 that a triangulation $\mathcal{T}$ of a region Π_k, corresponds to a cluster tilting set in $\mathcal{C}_n$, and applying $\rho^i(\mathcal{T})$, for $0 \leq i \leq p-1$, gives a cluster tilting set of $\mathcal{C}_{n,p}$, by the previous argument. □

Proposition 5.5 gives a geometric/combinatorial analogue to [Zhu08, Thm: 3.3].

Corollary 5.6. *The number of non isomorphic direct summands of a basic cluster tilting object in $\mathcal{C}_{n,p}$ is pn.*

Proof. Since each region Π_k is homotopic to a regular $n+3$-gon, the set $\mathcal{T}$ consists of n diagonals, hence by the previous result we deduce that the cardinality of a cluster tilting set $\mathcal{T}'$ in $\mathcal{C}_{n,p}$ is pn. □

5.1 Complements of almost complete basic cluster tilting objects

The counterpart to the exchange of cluster variables intrinsic in the definition of cluster algebras is given by the study of complements of *almost complete basic tilting objects*. By the last we mean a basic cluster tilting object from which a direct summand has been removed. Then one looks for all possible complementary objects such that the direct sum is again a cluster tilting object.

Given a tilting object $T = \bigoplus T_k$ in a cluster category, it is known that one can replace an indecomposable T_k of T by exactly one other indecomposable object. This is a consequence of the fact that the cluster category is 2-CY. Whereas in the module category of a finite dimensional hereditary algebra one can perform such a replacement in at most one different way. Here we analyse the behaviour in a fractionally CY-category.

Proposition 5.7. *In the repetitive cluster category $\mathcal{C}_{n,p}$, there are exactly 2 complements to an almost complete basic cluster tilting object.*

Proof. Assume that a triangulation of Π^p as in Proposition 5.5 is given. Then we observe that if a diagonal D_X is removed from the triangulation, we are left with a region delimited by four sides. This rectangular region has all its vertices on the circle which inscribes Π^p. Hence we are in the conditions of the Ptolemy relation and we can replace D_X with a unique other diagonal $D_{X'}$ completing again the triangulation. Then the triangulation obtained in this way is not as the one given in Proposition 5.5, so it does not give rise to a cluster tilting object of $\mathcal{C}_{n,p}$. Thus we have to perform the same exchange in each region Π_k, for $1 \leq k \leq p$, in order to obtain a new cluster tilting object in $\mathcal{C}_{n,p}$. Hence the claim follows. □

6 Repetitive higher cluster categories $\mathcal{C}^m_{n-1,p}$

In this section we study orbit categories of the form

$$\mathcal{C}^m_{n-1,p} := \mathcal{D}/\langle(\tau^{-1}[m])^p\rangle.$$

We call them *repetitive higher cluster categories of type A_{n-1}*. The class of objects is given by $\tau^{-p}[pm]$-orbits of objects in $\mathcal{D}$ and the space of morphisms is given by

$$\mathrm{Hom}_{\mathcal{C}^m_{n-1,p}}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, (\tau^{-p}[pm])^i Y).$$

Remark 6.1. *The category $\mathcal{C}^m_{n-1}$ is a subcategory of $\mathcal{C}^m_{n-1,p}$ which is usually not full.*

We will see below that these categories are interesting because their geometric models provides us with a geometric description of the bounded derived category $\mathcal{D}$.

Repetitive higher cluster categories have similar properties to the repetitive cluster categories. In particular, we can prove:

Lemma 6.2. *For all $m \in \mathbb{N}$.*

1. *The projection functor $\pi^m_p : \mathcal{D} \to \mathcal{C}^m_{n-1,p}$ is a triangle functor.*
2. *The category $\mathcal{C}^m_{n-1,p}$ is triangulated, with Serre functor $\tau[1]$ induced from $\mathcal{D}$.*
3. *$\mathcal{C}^m_{n-1,p}$ is fractionally Calabi Yau of dimension $\frac{p(m+1)}{p}$ and Krull-Schmidt.*
4. *$\mathrm{Ind}(\mathcal{C}^m_{n-1,p}) = \bigcup_{i=0}^{p-1}(\mathrm{Ind}(\tau^{-p}[pm])^i\mathcal{C}^m)$.*

Proof. The first two claims follow from [Kel08, Theorem 1] and [BMR+06, Proposition 1.3]. Concerning the fractionally Calabi-Yau dimension, one computes easily that $(\tau[1])^p \cong [p(m+1)]$ in $\mathcal{C}^m_{n-1,p}$. The last claim can be deduced from a similar reasoning as in [BMR+06, Proposition 1.6]. $\square$

6.1 $\mathcal{C}^m_{n-1,p}$ via m-diagonals in Π^p

In the following we present a geometric model for the category $\mathcal{C}^m_{n-1,p}$ which arises as a generalization of the one given for $\mathcal{C}_{n,p}$. Indecomposable objects are modelled by a subset of diagonals in Π^p and irreducible morphisms will be associated to rotations of such modulo the mesh relations.

Let Π^p be a regular $p(nm+1)$-gon, $m, n \in \mathbb{N}$. Then we divide Π^p into regions as we did in Section 3.1 for $m = 1$ and where n is replace by $n-1$. Then each region Π_k in Π^p is homotopic to a regular $(nm+2)$-gon. See also Figure 2.

For the convenience of the reader we recall that an *m-diagonal in a regular $(nm+2)$-gon*, is a diagonal which divides the polygon into two parts, each having 2 vertices $\mod m$.

Observe that m-diagonals have been studied by many authors, see [BM08], [BM07], [Tza06], [Tho07], [Tor11], [BT09], [FR05], ...

We will now adapt the notion of m-diagonals to the case at hand.

Definition 6.3. *The* m-diagonals of Π^p *are given by the union of the m-diagonals in each region Π_k, for $1 \leq k \leq p$.*

Notation 6.4. *In the writing of a diagonal (i, j, k) in Π^p, we consider the operations on the first two indices modulo m, and the last one modulo p. Remember that the last index specifies the region Π_k in Π^p. As every where else in the this paper we assume that $i < j$.*

We associate a quiver to the m-diagonals in Π^p as follows.

Definition 6.5. *Let $\Gamma^m_{n-1,p}$ be the* quiver of m-diagonals of Π^p*. Its vertices are the m-diagonals, and the arrows are given as follows.*

1. *If $j \neq N-(m-1)$,*

$$\begin{tikzcd}[row sep=small] & (i, j+m, k) \\ (i, j, k) \arrow[ur] \arrow[dr] & \\ & (i-m, j, k) \end{tikzcd}$$

2. *If $j = N-(m-1)$ then $(i, N-(m-1), k) \to (i-m, N-(m-1), k)$.*

3. *Furthermore, $(i, N-(m-1), k) \to (1, i, k+1)$.*

Observe that the description 1. and 2. corresponds to an irreducible m-rotations in Π^p as defined in [BM08]. That is an irreducible clockwise rotations between m-diagonals around the vertex i or j inside a region Π_k. We denote the set of these operations by IrrRot_m. The third type describes the composition of an m-irreducible rotation inside Π_k around the vertex i with the clockwise rotation $\rho(1, i, k) = (1, i, k+1)$. We denote the second type of operation by $\mathrm{Irr}\rho\mathrm{Rot}_m$.

Then we equip $\Gamma^m_{n-1,p}$ with a translation map.

The first part describes an anticlockwise rotation around the center of Π_k through $\frac{2m\pi}{nm+2}$, the second defines a composition of a rotation as in the first case, together with with an anticlockwise rotation around the center of Π^p through $\frac{2\pi}{p}$

Definition 6.6. *The* translation τ_m *maps*

- (i, j, k) *to* $(i - m, j - m, k)$ *if* $i, j \neq 1$.
- *If* $i = 1$

$$\tau_m(1, j, k) = \rho^{-1}(1 - m, j - m, k) = (1 - m, j - m, k - 1).$$

The proof of the next result is straightforward. We use the fact that the AR-quiver of $\mathcal{C}^m_{n-1}$ is isomorphic to the quiver of m-diagonals of Π as shown in [BM08, Proposition 5.4], then we use similar arguments as for Lemma 3.5

Lemma 6.7. *There is an isomorphism of stable translation quivers between the AR-quiver of* $\mathcal{C}^m_{n-1,p}$ *and* $\Gamma^m_{n-1,p}$.

6.2 Tilting theory for $\mathcal{C}^m_{n-1,p}$

In this section we determine the m-cluster tilting objects of $\mathcal{C}^m_{n-1,p}$ geometrically. In the following let $\mathcal{K} = \mathcal{D}, \mathcal{C}^m_{n-1}$ or $\mathcal{C}^m_{n-1,p}$.

Definition 6.8. *An* m-cluster tilting set $\mathcal{T}$ of $\mathcal{C}^m_{n-1,p}$ *is a maximal collection of pairwise non isomorphic indecomposable objects* $T_1, \ldots, T_r \in \mathcal{K}$, *such that*

$$\mathrm{Ext}^i_{\mathcal{K}}(T_j, X) = 0 = \mathrm{Ext}^i_{\mathcal{K}}(X, T_j), \quad \textit{for } 1 \leq i \leq m, \textit{ and } 1 \leq j \leq r,$$

if and only if $X \in \mathcal{T}$. *An* m-cluster tilting object T *in* $\mathcal{K}$ *is the direct sum of the objects* $T_1, \ldots, T_r$ *of an* m*-cluster tilting set of* $\mathcal{K}$.

Definition 6.9. *An* m-angulation *of a regular* $nm+2$*-gon consists of a maximal collection of non crossing* m*-diagonals in the polygon. Similarly an* m-angulation of Π^p *consists of a maximal set of non crossing* m*-diagonals in* Π^p.

For higher cluster categories the following result is known. It follows from [Tho07, Theorem 1] combined with [BM08, Section 4.].

Lemma 6.10. *Any* m*-cluster tilting object in* $\mathcal{C}^m_{n-1}$ *corresponds to the formal direct sum of elements in an* m*-angulations of a regular* $nm+2$*-gon.*

Thus, an m-cluster tilting objects in $\mathcal{C}^m_{n-1}$ has $n-1$ non isomorphic summands.

Definition 6.11. *An object* $T \in \mathcal{K}$ *is called* maximal m-rigid *if it is a direct sum of non isomorphic indecomposable objects* $T_1, \ldots, T_t$ *such that* $\mathrm{Ext}^i_{\mathcal{K}}(T_j, T_k) =$ *for all* $1 \leq i \leq m$, *and* $1 \leq j, k \leq t$, *and maximal with respect to this property.*

The following useful result is due to A. Wrålsen,[Wra09].

Lemma 6.12. *Any* m*-cluster tilting object in* $\mathcal{C}^m_{n-1}$ *is induced by maximal* m*-rigid object in the fundamental domain of* $\tau^{-1}[m]$ *in* $\mathcal{D}$.

Proposition 6.13. *Cluster tilting sets $\mathcal{T}'$ in $\mathcal{C}^m_{n-1,p}$ correspond bijectively to*

$$\mathcal{T}' = \mathcal{T} \cup \rho(\mathcal{T}) \cup \dots \rho^{p-1}(\mathcal{T})$$

where $\mathcal{T}$ is an m-angulation of the region Π_1 in Π^p.

Proof. First let $\mathcal{T}$ be an m-angulation of a given region Π_1 in Π^p. By Lemma 6.10 it corresponds to a cluster tilting object $T_{\mathcal{C}^m_{n-1}}$ of $\mathcal{C}^m_{n-1}$. By Lemma 6.12 it can be lifted to a maximal m-rigid object $T_{\mathcal{D}}$ of $\mathcal{D}$ contained in a fundamental domain of $\tau^{-1}[m]$. Consider its $(\tau^{-1}[m])$-orbit $\{(\tau^{-1}[m])^i(T_{\mathcal{D}}) | i \in \mathbb{Z}\}$ in $\mathcal{D}$ which we denote again by $T_{\mathcal{D}}$. Let $T_{\mathcal{C}^m_{n-1,p}}$ be the projection of $T_{\mathcal{D}}$ under the functor $\pi^m_p : \mathcal{D} \to \mathcal{C}^m_{n-1,p}$. Then $T_{\mathcal{C}^m_{n-1,p}}$ is a direct sum of pairwise non isomorphic indecomposable objects, which we denote by $T_1, \dots, T_t$ and we compute:

$$\begin{aligned}
\mathrm{Ext}^i_{\mathcal{C}^m_{n-1,p}}(T_k, T_l) &= \mathrm{Hom}_{\mathcal{C}^m_{n-1,p}}(T_k, T_l[i]) \\
&= \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T_k, (\tau^{-p}[mp])^j T_l[i]) \\
&= \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}^i_{\mathcal{D}}(T_k, (\tau^{-pj}[mpj]) T_l) \\
&= 0
\end{aligned}$$

for all T_k, T_l, $1 \leq k, l \leq t$ and $1 \leq i \leq m$. In fact, the first and third equality hold as the categories we consider are triangulated. The second follows from the definition of morphisms in $\mathcal{C}^m_{n-1,p}$. The last equality follows because if $T_{\mathcal{D}}$ is m-rigid then the same is true for $(\tau^{-1}[m])(T_{\mathcal{D}})$, hence also for $(\tau^{-j}[mj])T_{\mathcal{D}}$. And as $T_{\mathcal{D}}$ is maximal, it follows that in the fundamental domain of $(\tau^{-1}[m])$, $\mathrm{Ext}^i_{\mathcal{D}}(T_k, T_l) = 0$ holds.

Now we show that $T_{\mathcal{C}^m_{n-1,p}}$ is also maximal with respect to this property. By Lemma 6.10 the objects $T_1, \dots, T_t$ of $\mathcal{C}^m_{n-1,p}$ correspond to a set $\mathcal{T}'$ of m-diagonals in Π^p. Furthermore, by construction this set is given by the rotations of $\mathcal{T}$ under ρ. If $\mathcal{T}'$ is not maximal, there must be a region Π_l in Π^p where the set of m-diagonals is not maximal, but this contradicts the fact that $\mathcal{T}$ was an m-angulation. This proves one direction of the claim.

For the other direction, assume we have an m-angulation $\mathcal{T}'$ of Π^p, and that $\mathcal{T}'$ is not of the claimed form. Then $\mathcal{T}'$ is such that there are regions Π_k and Π_l in Π^p with m-angulations that are not obtained one from the other under ρ. Denote these two different m-angulations by $\mathcal{T}_k$ and $\mathcal{T}_l$. By Lemma 6.10 they correspond to m-cluster tilting objects in $\mathcal{C}^m_{n-1}$.

Consider the projection functor $\mu_p : \Pi^p \to \Pi$ sending a diagonal D_X of Π^p to the corresponding diagonal D_X in Π, and a morphism between D_X and D_Y in Π^p to the corresponding morphism between D_X and D_Y in Π.

Then $\mu_p(\mathcal{T}_k)$ does not coincide with $\mu_p(\mathcal{T}_l)$ as there is no $1 \leq i \leq p$ such that $\rho^i(\mathcal{T}_k) = \mathcal{T}_l$. Hence, there are two m-diagonals, say D_V and D_W that cross in Π, because otherwise the two m-angulations $\mathcal{T}_k)$ and $\mathcal{T}_l$ wouldn't be maximal. Therefore, we conclude by Lemma 6.10 that the corresponding objects V and W in $\mathcal{C}^m_{n-1}$ are such that $\dim(\mathrm{Ext}^i_{\mathcal{C}^m_{n-1}}(V, W)) > 0$ for an $1 \leq i \leq m$. Then it follows that also $\dim(\mathrm{Ext}^i_{\mathcal{C}^m_{n-1,p}}(V, W)) > 0$, as $\mathcal{C}^m_{n-1}$ is a subcategory of $\mathcal{C}^m_{n-1,p}$.

Hence the objects in $\mathcal{C}^m_{n-1,p}$ corresponding to the diagonals $\mathcal{T}'$ cannot form an m-cluster tilting set. $\square$

With this result we deduce immediately the following.

Corollary 6.14. *Any m-cluster tilting set in $\mathcal{C}^m_{n-1,p}$ contains $p(n-1)$ non isomorphic objects.*

6.3 Complements of almost complete m-cluster tilting objects in $\mathcal{C}^{\Updownarrow}{}_{n-1,p}$

An *almost complete m-cluster tilting object* in $\mathcal{C}^m_{n-1,p}$ is an m-cluster tilting object with one indecomposable summand removed. For $p = 1$ complements of almost complete m-cluster tilting objects where studied in [Tho07], [ZZ09] and [Wra09].

Observe that $m + 1$ is the CY dimension of $\mathcal{C}^m_{n-1,p}$, for $p = 1$.

Proposition 6.15. *In $\mathcal{C}^m_{n-1,p}$ there are exactly $(m + 1)$ complements to an almost complete m-cluster tilting object.*

Proof. For $p > 1$, one can proceed as in Proposition 5.7 together with the result of Proposition 6.13. □

7 Geometric model for $\mathcal{D}$

In this section we model geometrically the bounded derived category of $\mathrm{mod}kA_n$. Our strategy is to adapt the construction of $\mathcal{C}^m_{n-1,p}$ given in the previous section. One difference is that we no longer associate indecomposable objects of the category at hand to m-diagonals but to their complements. More precisely, we consider the diagonals in a polygon which are not 2-diagonals. The other difference is that the polygon we consider here has no longer finitely many vertices.

Complements to m-diagonals in a given polygon Π have been studied in [Lam11]. The result we need for the modelling of $\mathcal{D}$ is Lemma 7.1 which is obtained from Corollary 4.13 and Theorem 4.16 in [Lam11]. The result in based on the notion of the m-th power of a translation quiver introduced in [BM08].

7.1 The m-th power of a translation quiver.

The *m-th power of* (Γ, τ), is a translation quiver (Γ^m, τ^m) which has the same vertices as Γ, and whose arrows are given by paths of length m in Γ: $(x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y)$, such that whenever τx_{i+1} is defined $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m-1$. Furthermore, the translation is given as $\tau^m := \tau \circ \cdots \circ \tau$, m-times. If Γ is the AR-quiver of $\mathcal{C}_{nm-1}$, the m-th power gives rise to the AR-quiver of $\mathcal{C}^m_{n-1}$, see [BM08, Theorem 7.2].

Lemma 7.1. *Let Π be a regular polygon with $nm + 2$ vertices, and $\Gamma_{A_{mn-1}}$ the quiver of its diagonals. Then we have*

$$(\Gamma_{A_{mn-1}})^2 = \Gamma^2_{n-1} \sqcup \Gamma_1 \sqcup \Gamma_2,$$

where $\Gamma_1 \cong \Gamma_2 \cong AR(\mathcal{D}^b(\mathrm{mod}kA_n)/[1])$, and $[1]$ is the shift functor on $\mathcal{D}^b(\mathrm{mod}kA_n)$.

As a consequence of this result it was observed in [Lam11] that the quivers of diagonals Γ_1 and Γ_2 have as vertices diagonals between pairs of vertices in Π of the form $(even, even)$ or (odd, odd) only.

7.2 The ∞-gon $\Pi^{\pm\infty}$

As next we extend the construction of the regular $p(2n+1)$-gon Π^p to a geometric figure with infinitely many sides. One way to do this is the following. We first define the polygon $\Pi^{\pm p}$, and we then let $p \to \pm\infty$.

Let Π_1 be a $N := (2n+2)$-gon and assume that its vertices lie all on a line (c.f. Figure 4). Then define the translation ϱ that shifts the region Π_1 to the next region to the right, and set $\varrho^k(\Pi_1) = \Pi_{k+1}$, for $-p \leq k \leq p$, then $\varrho^{-k}\varrho^k = \varrho^k\varrho^{-k} = Id$.

In this way we obtain a figure with $2p+1$ regions homotopic to Π_1, and such that the first vertex of Π_{k+1} corresponds to the last of Π_k. Denote this figure by $\Pi^{\pm p}$. The diagonals in $\Pi^{\pm p}$ we are considering are the diagonals connecting even numbered vertices within a region Π_k, for any k between $-p$ and p. We call these diagonals 2^c-*diagonals* Observe that the 2^c-diagonals in Π_1 are the vertices of the quiverΓ_1 of Lemma 7.1. We furthermore remark that for our purpose, we might as well have chosen the diagonals connecting odd vertices.

7.3 The quiver $\Gamma^{\mathbf{even}}_{n,\pm\infty}$

We define a translation quiver on the 2^c-diagonals. The arrows between them are defined in a similar way as in Section 6. Let (i,j,k) be a 2^c-diagonal in a region Π_k of $\Pi^{\pm p}$ with $j \neq N$, then

$$\begin{tikzcd} & (1,j+2,k) \\ (i,j,k) \arrow[ur] \arrow[dr] & \\ & (i-2,j,k) \end{tikzcd}$$

where the first two entrances are understood modulo 2, and $-p \leq k \leq p$ modulo p. Furthermore, for all $k \neq \pm p$ we have arrows

$$\begin{tikzcd} & (2,i,k+1) \\ (i,N,k) \arrow[ur] \arrow[dr] & \\ & (i-2,N,k) \end{tikzcd}$$

where $(2,i,k+1) = \varrho(2,i,k+1)$. The condition on k is needed because we want to avoid that there is an arrow linking the diagonals of the region Π_p to those of the region Π_{-p}. We will refer to the arrows $(i,N,k) \to (2,i,k+1)$, as *connecting arrows.*

Letting $p \to \infty$ we obtain infinitely many regions homotopic to Π_1, and we call the corresponding figure $\Pi^{\pm\infty}$.

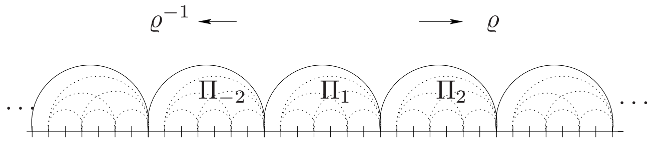

Figure 4: $\Pi^{\pm\infty}$ as model for $\mathcal{D}^b(\mathrm{mod}kA_3)$.

We denote the quiver of 2^c-diagonals of $\Pi^{\pm\infty}$ that we just defined by $\Gamma^{\mathrm{even}}_{n,\pm\infty}$.

7.4 Translation map for $\Gamma^{\mathbf{even}}_{n,\pm\infty}$

We equip $\Gamma^{\text{even}}_{n,\infty}$ with the translation τ_2 from Definition 6.6 and letting p tend to infinity. That is, $\tau_2(i,j,k) = (i-2, j-2, k)$ for all i, j different from 2, and $\tau_2(2,j,k) = \varrho^{-1}(N, j-2, k) = (j-2, N, k-1)$. It can be proven as in Lemma 3.5 that this defines a stable translation quiver, and taking the mesh category of it specifies a rotation rule for the 2^c-diagonals of $\Pi^{\pm\infty}$ arising from the mesh relations of $\Gamma^{\text{even}}_{n,\infty}$.

7.5 Geometric model of $\text{mod}kA_n$

We focus on the smaller polygon in $\Pi^{\pm\infty}$, say on say Π_1. Its 2^c-diagonals together with the rotations between them defined above, restricted to the region Π_1 give rise to a subquiver $\Gamma^{\text{even}}_{\Pi_1}$ of $\Gamma^{\text{even}}_{n,\pm p}$, with translation given by restricting τ_2 to Π_1. As we are considering only the region Π_1, it follows from the definition that we do not allow rotations from the 2^c-diagonal $(i, N, 1)$ around the vertex N, as these would be linked with an arrow to a 2^c-diagonal in the neighboring region Π_2.

For an example, we invite to compare with Figure 4. In this case Π_1 is an octagon and the dotted lines are all the 2^c-diagonals of Π_1.

We are now ready to deduce the next result which shows Π_1 is a geometric model for the category $\text{mod}kA_n$.

We assume that the path algebra kA_n is taken over an equioriented Dynkin quiver of type A_n.

Lemma 7.2. *There is an isomorphism of stable translation quivers*

$$\Gamma^{even}_{\Pi_1} \cong AR(\text{mod}kA_n).$$

Proof. By construction Π_1 is homotopic to a regular $2n+2$-gon. Hence, the number of 2^c-diagonals in Π_1 is $\frac{n(n+1)}{2}$. This number agrees with the isomorphism classes of indecomposable objects in $\text{mod}kA_n$. Using the correspondence between indecomposable objects in $\text{mod}kA_n$ and positive roots of the associated Dynkin diagram (Gabriels Theorem), we associate positive roots to the 2^c-diagonals of Π_1 and indecomposable modules to the positive roots.

Given a 2^c-diagonal $(i, j, 1)$ we let $|i-j|$ be its size. Then we associate the 2^c-diagonals of size 2 to the simple roots α_i, $1 \le i \le n$ of the associated Dynkin diagram. As the length of the segment augments, we increase the number of summands of the positive root. Remark that the longest 2^c-diagonal of Π_1 is $(2, 2n+2)$, and it corresponds to $\alpha_1 + \cdots + \alpha_n$.

Due to the shape of $\Gamma^{\text{even}}_{\Pi_1}$ and the well known shape of $AR(\text{mod}kA_n)$, it is clear that the map we just defined gives rise to an isomorphism of quivers, and it easy to check that it preserves the translation so that it becomes an isomorphism of stable translation quivers. □

7.6 Geometric model of the bounded derived category

We use a well known result of D. Happel stating that the AR-quiver of $\mathcal{D}$ is isomorphic to $\mathbb{Z}A_n$, with the usual translation τ to link the AR-quiver of $\mathcal{D}$ to the quiver $\Gamma^{\text{even}}_{n,\pm\infty}$ of 2^c-diagonals in $\Pi^{\pm\infty}$. By taking the mesh category of $(\Gamma^{\text{even}}_{n,\pm\infty}, \tau_2)$ we thus obtain a geometric model of $\mathcal{D}$.

Proposition 7.3. *There is an isomorphism of stable translation quivers*

$$(\Gamma^{even}_{n,\pm\infty}, \tau_2) \xrightarrow{\sim} (\mathbb{Z}A_n, \tau).$$

Proof. By Lemma 7.2 we know that $\Gamma^{\text{even}}_{\Pi_1}$ is isomorphic to $AR(\text{mod}kA_n)$. Furthermore, we observe that by shifting around copies of Π_1, the translation ϱ reproduces copies of $\Gamma^{\text{even}}_{\Pi_1}$ in $\Gamma^{\text{even}}_{n,\pm\infty}$. Thus, it remains to study the connecting arrows. In $\Gamma^{\text{even}}_{n,\pm\infty}$ these are the arrows between 2^c-diagonals of the form (i, N, k) and $(2, i-2, k+1)$ in the different regions Π_k and Π_{k+1} inside $\Pi^{\pm\infty}$. By construction these arrows are exactly the connecting arrows between different copies of the AR-quiver of modkA_n in $AR(\mathcal{D})$. □

Corollary 7.4. *There is a morphism of functors between ϱ in the category of 2^c-diagonals of $\Pi^{\pm\infty}$ and the shift functor $[1]$ on $\mathcal{D}$.*

Proof. This follows directly from the arguments given in the proof of the previous result. □

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